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Distributive Lattices of λ-simple Semirings

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Abstract. In this paper, we study the decomposition of semirings with a semilattice additive reduct. For, we introduce the notion of principal left k-radicals $\Lambda(a) = \{x \in S \mid a \overset{l}{\longrightarrow}^{\infty} x\}$ induced by the transitive closure $\overset{l}{\longrightarrow}^{\infty}$ of the relation $\overset{l}{\longrightarrow}$ which induce the equivalence relation λ. Again non-transitivity of $\overset{l}{\longrightarrow}$ yields an expanding family $\{\overset{l}{\longrightarrow}^{n}\}$ of binary relations which associate subsets $\Lambda_n(a)$ for all $a \in S$, which again induces an equivalence relation λ_n. We also define $\lambda(\lambda_n)$-simple semirings, and characterize the semirings which are distributive lattices of $\lambda(\lambda_n)$-simple semirings.

Keywords: Principal left k-radical, Distributive lattice congruence, Completely semiprime k-ideal, λ-simple semiring, Distributive lattice decomposition.

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1. Introduction

The notion of semirings was introduced by Vandiver [12]. Semiring is a generalization of both an associative ring as well as of distributive lattices. Since semiring is a $(2, 2)$ algebra, it has many applications in different areas of

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mathematics, idempotent analysis, physics, computer science etc. The underlying semirings in idempotent analysis, syntactic semirings, Max-plus algebra, Kleene algebra are those whose additive reduct is a semilattice, i.e., idempotent and commutative. On the other hand the structure of semirings with semilattice additive reduct have been studied by Bhuniya and Mondal in [1, 2, 3, 4], Mondal [9], Mondal and Hansda [10], Mondal and Bhuniya [11]. Distributive lattice decomposition of such semirings is one of the most beautiful technique in the study of structure of semirings. In [2], Bhuniya and Mondal gave the description of the least distributive lattice congruence on a semiring in three different ways, where different descriptions produced different types of simpler structures. All these works are motivated by the idea of the semilattice decompositions of the semigroups through the least semilattice congruence given by A.H. Clifford [6]. That has been an elegant technique to give the description of the structure of different classes of semigroups. In our work we generalize the idea of semilattice decomposition of semigroups to distributive lattice decomposition of semirings.

This paper is a continuation of our study on the structure of semirings in $\mathcal{SL}^+$ [2]. Our main aim is to decompose the semirings with semilattice additive reduct through the least distributive lattice congruence into simpler components. The preliminaries and prerequisites are given in section 2, and state some results from [2, 3]. In section 3, we introduce the notion of principal left k-radicals $\Lambda(a) = \{x \in S \mid a \overset{l}{\longrightarrow}^{\infty} x\}$ induced by the transitive closure $\overset{l}{\longrightarrow}^{\infty}$ of the relation $\overset{l}{\longrightarrow}$, give its some basic characteristics, define an equivalence relation λ induced by the principal left k-radicals. Again non-transitivity of $\overset{l}{\longrightarrow}$ leads to give an expanding family $\overset{l}{\longrightarrow}^{n}$ of binary relations which associate subsets $\Lambda_n(a)$ for all $a \in S$, which induces equivalence relation λ_n. We also define $\lambda(\lambda_n)$-simplicity of a semiring , and characterize the semirings which are distributive lattices of λ-simple semirings. Finally we give the characterization of the semirings which are distributive lattices of λ_n-simple semirings.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both $(S, +)$ and $(S, \cdot)$ are semigroups and such that the following distributive laws hold: for $x, y, z \in S$,

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

Any distributive lattice D is a semiring $(D, +, \cdot)$ such that both the additive reduct $(D, +)$ and the multiplicative reduct $(D, \cdot)$ are semilattices together with the absorptive law:

$$x + x.y = x \text{ for all } x, y \in S.$$

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Thus the semiring is a generalization of both rings and distributive lattices. By $\mathcal{SL}^+$ we denote the category of all semirings $(S,+,\cdot)$ with a semilattice additive reduct. Throughout this paper, unless otherwise stated, S is always a semiring in $\mathcal{SL}^+$.

Let A be non-empty subset of a semiring S. Then the *k-closure* of A is defined by $\overline{A} = \{x \in S \mid x + a_1 = a_2$, for some $a_1, a_2 \in A\}$, and the *k-radical* of A by $\sqrt{A} = \{x \in S \mid (\exists\ n \in \mathbb{N})\ x^n \in \overline{A}\}$. Then $\overline{A} \subseteq \sqrt{A}$ by definition, and $A \subseteq \overline{A}$, since $(S,+)$ is a semilattice. Again $x + a_1 = a_2$ implies $x+a_2 = x+x+a_1 = x+a_1 = a_2$. So, one can also write $\overline{A} = \{x \in S \mid x+a = a$ for some $a \in A\}$. An ideal I of a semiring S is a *k-ideal* if $I = \overline{I}$. A nonempty subset A of S is called *completely semiprime* if for $x \in S, x^2 \in A$ implies $x \in A$. It can easily be checked that k-ideal I is completely semiprime if and only if $\sqrt{I} = I$.

An equivalence relation ρ on a semiring S is called a *left congruence* if for all $a, b, c \in S,\ a\rho b$ implies that $(a+c)\rho(b+c)$ and $ca\rho cb$. The *right congruences* are defined dually. An equivalence relation ρ on S is called a *congruence* if it is both a left and a right congruence on S. A congruence ρ on S is called a *distributive lattice congruence* on S if the quotient semiring S/ρ is a distributive lattice. If $\mathcal{C}$ is a class of semirings we refer to semirings in $\mathcal{C}$ as $\mathcal{C}$-semirings. A semiring S is called a *distributive lattice of $\mathcal{C}$-semirings* if there exists a congruence ρ on S such that S/ρ is a distributive lattice and each ρ-class is a semiring in $\mathcal{C}$.

Lemma 2.1 ([3], 2.1)**.** *Let S be a semiring.*
(a) *For $a, b \in S$ the following statements are equivalent:*
(i) there are $s_i, t_i \in S$ such that $b + s_1at_1 = s_2at_2$;
(ii) there are $s, t \in S$ such that $b + sat = sat$;
(iii) there is $x \in S$ such that $b + xax = xax$.
(b) *If $a, b, c \in S$ are such that $b + xax = xax$ and $c + yay = yay$ for some $x, y \in S$, then there is $z \in S$ such that $b + zaz = zaz = c + zaz$.*
(c) *If $a, b, c \in S$ are such that $c + xax = xax$ and $c + yby = yby$ for some $x, y \in S$, then there is $z \in S$ such that $c + zaz = zaz$ and $c + zbz = zbz$.*

Lemma 2.2 ([3], 2.2)**.** *For a semiring S and $a, b \in S$ the following statements hold.*

(1) $\overline{SaS}$ *is a k-ideal of S.*
(2) $\sqrt{SaS} = \sqrt{\overline{SaS}}$.
(3) $b^m \in \sqrt{SaS}$ *for some $m \in \mathbb{N} \Leftrightarrow b^k \in \sqrt{SaS}$ for all $k \in \mathbb{N}$.*

In [3], the authors studied the structure of the semirings in $\mathcal{SL}^+$, and during this, the description of the least distributive lattice congruence η on a semiring S was given, where $\longrightarrow^\infty = \bigcup_{n=1}^\infty \longrightarrow^n$ is the transitive closure of $\longrightarrow$ and $\eta = \longrightarrow^\infty \cap (\longrightarrow^\infty)^{-1}$[2]. In [2], for the second description of the least distributive lattice congruence on a semiring S, Bhuniya and Mondal introduced the set $M(a)$, for each $a \in S$ defined by

$$M(a) = \{x \in S \mid a \longrightarrow^{\infty} x\}.$$

There, as one of the most important basic characteristics of the set $M(a)$, it was shown that $M(a)$ is the least completely semiprime k-ideal of S containing a, and an equivalence relation on S was determined in respect of producing the same principal completely semiprime k-ideal denoted by $\mathcal{M}$, and was given by: for $a, b \in S$,

$$a\mathcal{M}b \Leftrightarrow M(a) = M(b).$$

Theorem 2.3 (3.6, [2])**.** *Let S be a semiring. Then $\mathcal{M}$ is the least distributive lattice congruence on S.*

Definition 2.4 (4.1, [2])**.** Let ρ be a binary relation on a semiring S. Then S is said to be *ρ-simple* if $\rho = S \times S$.

Thus a semiring S is *$\mathcal{M}$-simple* if $\mathcal{M} = S \times S$. Again $\mathcal{M}$ being the least distributive lattice congruence, if S is $\mathcal{M}$-simple then there will be no other distributive lattice congruence on S except the universal relation $S \times S$. In the following theorem characterization of such semirings was given, where we state only four equivalent conditions.

Theorem 2.5 (4.2, [2])**.** *The following conditions on a semiring S are equivalent:*

(1) *$\omega = S \times S$ is the only distributive lattice congruence on S;*
(1) *S is η-simple;*
(1) *S is $\mathcal{M}$-simple;*
(1) *$M(a) = S$ for all $a \in S$.*

Definition 2.6 (4.3, [2])**.** A semiring S is said to be *indecomposable* if the universal relation ω is the only distributive lattice congruence on S.

Theorem 2.7 (4.5, [2])**.** *Every semiring S is a distributive lattice of indecomposable semirings.*

For undefined concepts in semigroup theory we refer to [8], for undefined concepts in semiring theory we refer to [7].

3. Principal left k-radicals and distributive lattices of λ-simple semirings:

In [9], the author introduced the following relations on a semiring S : for $a, b \in S$, $a \mid_l b$ if $b \in \overline{Sa}, a \xrightarrow{l} b$ if $a \mid_l b^n$ for some $n \in \mathbb{N}$. In this paper we define the following: in general, $\xrightarrow{l}$ is not transitive. Non-transitiveness of $\xrightarrow{l}$ then produces a family of binary relations $\xrightarrow{l}^{n}$ for each $n \in \mathbb{N}$. For $a, b \in S, a \xrightarrow{l}^{n+1} b$ if there exists $x \in S$ such that $a \xrightarrow{l}^{n} x \xrightarrow{l} b, n \in \mathbb{N}$ and $a \xrightarrow{l}^{\infty} b$ if $a \xrightarrow{l}^{n} b$ for some $n \in \mathbb{N}$.

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Lemma 3.1. *Let S be a semiring.*
(a) *For $a, b \in S$ the following statements are equivalent:*

(1) *there are $s_i \in S$ such that $b + s_1 a = s_2 a$;*

(2) *there are $s \in S$ such that $b + sa = sa$.*

(b) *If $a, b, c \in S$ such that $c + xa = xa$ and $d + yb = yb$ for some $x, y \in S$, then there is some $z \in S$ such that $c + za = za$ and $d + zb = zb$.*

Proof. (a) (1) $\Rightarrow$ (2) holds for $s = s_1 + s_2$, since $(S, +)$ is a semilattice. Other implication is clear.
(b) $z = x + y$ serves our purpose. □

Following the ideas of Ćirić and Bogdanović[5], we introduce the following: For every $a \in S$ and $n \in \mathbb{N}$

$$\Lambda(a) = \{x \in S \mid a \overset{l}{\longrightarrow}^{\infty} x\},\ \Lambda_n(a) = \{x \in S \mid a \overset{l}{\longrightarrow}^{n} x\}.$$

For every $a \in S$, $\Lambda(a)$ is called the *principal left k-radical* in S containing a. Here we present some basic characteristics of these sets.

Lemma 3.2. *Let S be a semiring and $a, b, c \in S$. Then*
(1) $\Lambda_1(a) = \sqrt{Sa}$.
(2) $\Lambda_n(a) \subseteq \Lambda_{n+1}(a) \subseteq \sqrt{S\Lambda_n(a)}, n \in \mathbb{N}$.
(3) $\Lambda(a) = \bigcup_{n \in \mathbb{N}} \Lambda_n(a)$.

Proof. (1) Let $x \in \Lambda_1(a)$. Then $a \overset{l}{\longrightarrow}^{1}$, i.e. $x \in \sqrt{Sa}$ so that $\Lambda_1(a) \subseteq \sqrt{Sa}$. If $y \in \sqrt{Sa}$, then $x^n \in \overline{Sa}$ for some $n \in \mathbb{N}$. This implies $a \overset{l}{\longrightarrow} x$, i.e. $x \in \Lambda_1(a)$ yielding $\sqrt{Sa} \subseteq \Lambda_1(a)$. Consequently, $\Lambda_1(a) = \sqrt{Sa}$.
(2) Let $x \in \Lambda_n(a)$. Then $a \overset{l}{\longrightarrow}^{n} x$, and $x \overset{l}{\longrightarrow} x$ together imply $a \overset{l}{\longrightarrow}^{n+1} x$. This yields $x \in \Lambda_{n+1}(a)$, whence $\Lambda_n(a) \subseteq \Lambda_{n+1}(a)$.

For the second inclusion, let $x \in \Lambda_{n+1}(a)$. Then $a \overset{l}{\longrightarrow}^{n+1} x$ so that $a \overset{l}{\longrightarrow}^{n} b \overset{l}{\longrightarrow} x$ for some $b \in S$. Now $a \overset{l}{\longrightarrow}^{n} b$ and $b \overset{l}{\longrightarrow} x$ imply $b \in \Lambda_n(a)$ and $x \in \sqrt{Sb}$ so that $x \in \sqrt{S\Lambda_n(a)}$. Thus one gets $\Lambda_{n+1}(a) \subseteq \sqrt{S\Lambda_n(a)}$.
(3) The proof is straight forward. □

Now we introduce two equivalence relations λ and λ_n on S by: for $a, b \in S$,

$$a\lambda b \Leftrightarrow \Lambda(a) = \Lambda(b) \text{ and } a\lambda_n b \Leftrightarrow \Lambda_n(a) = \Lambda_n(b).$$

These equivalences are generalizations of the well-known *Green's relation* $\overline{\mathcal{L}}$. A semiring S is said to be $\lambda(\lambda_n)$*-simple* if $\lambda(\lambda_n) = S \times S$. A semiring S is called a *distributive lattice(chains) of* $\lambda(\lambda_n)$*-simple semirings* if there exists a

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congruence ρ on S such that S/ρ is a distributive lattice(chain) and each ρ-class is a $\lambda(\lambda_n)$-simple semiring.

Lemma 3.3. *Suppose S is a distributive lattice $\mathcal{D}$ of subsemirings $S_\alpha; \alpha \in \mathcal{D}$.*

(1) *If $a \in S_\alpha, b \in S_\beta, \alpha, \beta \in \mathcal{D}$ are such that $a \stackrel{l}{\longrightarrow} b$, then $\beta \leq \alpha$.*

(2) *If $a, b \in S_\alpha,\ \alpha \in \mathcal{D}$, then $a \stackrel{l}{\longrightarrow}^n b$ in S implies that $a \stackrel{l}{\longrightarrow}^n b$ in S_α.*

Proof. Let ρ be a distributive lattice congruence on S so that S is a distributive lattice $\mathcal{D}$ of subsemirings $S_\alpha; \alpha \in \mathcal{D}$.
(1) Now $a \stackrel{l}{\longrightarrow} b$ implies $b^n + xa = xa$ for some $n \in \mathbb{N}, x \in S$, by Lemma 3.1. Then $(a+b)\rho(a + xa + b^n) = (a + xa)\rho a$, which gives $\alpha + \beta = \alpha$, i.e, $\beta \leq \alpha$.
(2) There are $x_i(i = 1, 2, ..., n-1)$ in S such that $a \stackrel{l}{\longrightarrow} x_1 \stackrel{l}{\longrightarrow} x_2 \stackrel{l}{\longrightarrow} ... \stackrel{l}{\longrightarrow} x_{n-1} \stackrel{l}{\longrightarrow} b$. Let $x_i \in S_{\beta_i}(i = 1, 2, ..., n-1), \beta \in D$. Then by (1) we get $\alpha \leq \beta_{n-1} \leq ... \leq \beta_2 \leq \beta_1 \leq \alpha$, and hence $\beta_i = \alpha$, i.e, $x_i \in S_\alpha$. Thus $a \stackrel{l}{\longrightarrow}^n b$ in S_α. □

Here we characterize the semirings which are distributive lattices of λ-simple semirings.

Theorem 3.4. *The following conditions are equivalent on a semiring S:*

(1) *S is a distributive lattice of λ-simple subsemirings;*
(2) *for all $a, b \in S, ab \in \Lambda(a)$;*
(3) *for every $a \in S, \Lambda(a)$ is the least completely semiprime k-ideal of S containing a;*
(4) *$\lambda = \eta$, the least distributive lattice congruence on S;*
(5) *for all $a, b \in S, b \in \overline{SaS}$ implies that $b \in \Lambda(a)$;*
(6) *for all $a, b \in S, \Lambda(ab) = \Lambda(a) \cap \Lambda(b)$.*

Proof. (1) ⇒ (2) Let S be a distributive lattice $\mathcal{D}$ of subsemirings $S_i; i \in \mathcal{D}$, and let $a, b \in S$. Then $ab, ba \in S_i$ for some $i \in \mathcal{D}$. Since S_i is λ-simple, $\Lambda(ab) = \Lambda(ba)$ in S_i, i.e. $ba \stackrel{l}{\longrightarrow}^\infty ab$ in S, and since $a \stackrel{l}{\longrightarrow} ba$ in S, $a \stackrel{l}{\longrightarrow}^\infty ab$ in S, i.e. $ab \in \Lambda(a)$.
(2) ⇒ (3) Let $x_1, x_2 \in \Lambda(a)$ and $s \in S$ be such that $s + x_1 = x_2$. Then $a \stackrel{l}{\longrightarrow}^\infty x_2 = (s + x_1) \stackrel{l}{\longrightarrow} s$ yields $s \in \Lambda(a)$. Thus $\Lambda(a)$ is a k-set. Let $x, y \in \Lambda(a)$. Then $a \stackrel{l}{\longrightarrow}^n x$ and $a \stackrel{l}{\longrightarrow}^n y$ for some $n \in \mathbb{N}$. Then by Lemma 3.1, there exist $m \in \mathbb{N}$ and $s, x_i, y_i(i = 1, 2, ..., n-1)$ in S such that $x_1^m + sa = sa, x_{i+1}^m + sx_i = sx_i(i = 1, 2, ..., n-2), x^m + sx_{n-1} = sx_{n-1}$ and $y_1^m + sa = sa, y_{i+1}^m + sy_i = sy_i(i = 1, 2, ..., n-1), y^m + sy_{n-1} = sy_{n-1}$. Multiplying both sides of $y^m + sy_{n-1} = sy_{n-1}$ by y on the right, we get $y^{m+1} + sy_{n-1}y = sy_{n-1}y$. Now by rearranging the terms we have $(x+y)^{m+1} = y^{m+1} + ux + xv + \sum_{i=1}^{k} u_i x v_i$ for some $u, v, u_i, v_i \in S$. Adding $sy_{n-1}y$ on both sides one gets $(x+y)^m + sy_{n-1}y = sy_{n-1}y + ux + xv + \sum_{i=1}^{k} u_i x v_i$. From this we write

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$(x+y)^{m+3}+(x+y)s(x+y_{n-1})y(x+y) = (x+y)s(x+y_{n-1})y(x+y)+(x+y)u(x+y_{n-1})(x+y)+(x+y)(x+y_{n-1})v(b+c)+\sum_{i=1}^{k}(x+y)u_i(x+y_{n-1})v_i(x+y)$. Now for $w = (x+y)s+y(x+y)+(x+y)u+v(x+y)+\sum_{i=1}^{k}(x+y)u_i+\sum_{i=1}^{k}v_i(x+y)+x+y$ we obtain $(x+y)^{m+2}+w(x+y_{n-1})w = w(x+y_{n-1})w$, which yields $(x+y_{n-1})w \overset{l}{\longrightarrow} (x+y)$. By hypothesis, $(x+y_{n-1}) \overset{l}{\longrightarrow} (x+y_{n-1})w$, so that $(x+y_{n-1}) \overset{l}{\longrightarrow}^{\infty} (x+y)$. Iterating this implication one gets $(x+a) \overset{l}{\longrightarrow}^{\infty} (x+y_1), (x+y_1) \overset{l}{\longrightarrow}^{\infty} (x+y_2), ..., (x+y_{n-2}) \overset{l}{\longrightarrow}^{\infty} (x+y_{n-1})$, and so $(x+a) \overset{l}{\longrightarrow}^{\infty} (x+y)$. Similarly $a \overset{l}{\longrightarrow}^{\infty} a$ and $a \overset{l}{\longrightarrow}^{\infty} x$ give $(a+a) \overset{l}{\longrightarrow}^{\infty} (a+x)$, i.e. $a \overset{l}{\longrightarrow}^{\infty} (a+x)$. Then by transitivity of $\overset{l}{\longrightarrow}^{\infty}$, we get $a \overset{l}{\longrightarrow}^{\infty} (x+y)$, that is, $x+y \in \Lambda(a)$. Let $x \in \Lambda(a)$ and $s \in S$. Then $a \overset{l}{\longrightarrow}^{\infty} x$ and since $x \overset{l}{\longrightarrow} sx$ and $x \overset{l}{\longrightarrow} xs$, by (2), so $xs, sx \in \Lambda(a)$. Let $x \in S$ such that $x^2 \in \Lambda(a)$. Then $a \overset{l}{\longrightarrow}^{\infty} x^2 \overset{l}{\longrightarrow} x$ implies $x \in \Lambda(a)$. Thus $\Lambda(a)$ is a completely semiprime k-ideal of S containing a. Let I be a completely semiprime k-ideal of S containing a. Then for $x \in \Lambda_1(a)$, one has $x^n + sa = sa$ for some $n \in \mathbb{N}$, so that $x^{n+1} + sax = sax \in I$, which implies $x \in \sqrt{I} = I$, since I is completely semiprime. Therefore $\Lambda_1(a) \subseteq I$. Assume that $\Lambda_n(a) \subseteq I$. Then $S\Lambda_n(a) \subseteq SI \subseteq I$, so $\Lambda_{n+1}(a) = \sqrt{S\Lambda_n(a)} \subseteq \sqrt{I} = I$. Hence by principle of mathematical induction $\Lambda(a) = \bigcup_{n\in\mathbb{N}} \Lambda_n(a) \subseteq I$. Hence $\Lambda(a)$ is the least completely semiprime k-ideal of S.

(3) $\Rightarrow$ (4) Since $\Lambda(a)$ is the least completely semiprime k-ideal of S, $\Lambda(a) = M(a)$, and so $\lambda = \eta$.

(4) $\Rightarrow$ (1) Follows from Theorems 2.5 and 2.7.

(1) $\Rightarrow$ (5) Let S be a distributive lattice $D = S/\rho$ of λ-simple semirings $L_\alpha = a\rho, \alpha \in D, a \in S$. Suppose $a, b \in S$ be such that $b \in \overline{SaS}$. Then $b + sas = sas$ for some $s \in S$, by Lemma 2.1. Now $sas\rho s^2 a\rho sa$ implies that $sas, sa \in L_\alpha$ for some $\alpha \in D$. Since L_α is λ-simple, $\Lambda(sas) = \Lambda(sa)$. Then $sa \overset{l}{\longrightarrow}^{\infty} sas$, and so there is $n \in \mathbb{N}$ such that $sa \overset{l}{\longrightarrow}^{n} sas$. Then by Lemma 3.1, there are $m \in \mathbb{N}$ and x, x_i, y_i in S such that $x_1^m + xsa = xsa, x_{i+1}^m + xx_i = xx_i(i = 1, 2, ..., n-2), (sas)^m + xx_{n-1} = xx_{n-1}$. Now $b^m + (sas)^m = (sas)^m$. Adding both sides by xx_{n-1}, one gets $b^m + ((sas)^m) + xx_{n-1}) = (sas)^m + xx_{n-1}$. This implies that $b^m + xx_{n-1} = xx_{n-1}$, and so $x_{n-1} \overset{l}{\longrightarrow} b$. Similarly from the remaining equalities we have $x_{n-2} \overset{l}{\longrightarrow} x_{n-1}, x_{n-3} \overset{l}{\longrightarrow} x_{n-2}, ..., x_1 \overset{l}{\longrightarrow} x_2, a \overset{l}{\longrightarrow} x_1$. From these one gets $a \overset{l}{\longrightarrow}^{\infty} b$. Thus $b \in \Lambda(a)$.

(5) $\Rightarrow$ (2) For $a, b \in S, (ab)^2 \in \overline{SaS}$ implies that $(ab)^2 \in \Lambda(a)$, that is, $a \overset{l}{\longrightarrow}^{\infty} (ab)^2$. Also $(ab)^2 \overset{l}{\longrightarrow} ab$ implies that $a \overset{l}{\longrightarrow}^{\infty} ab$. Thus $ab \in \Lambda(a)$.

(2) $\Rightarrow$ (6) Let $x \in \Lambda(ab)$. Then $ab \overset{l}{\longrightarrow}^{\infty} x$ so that $ab \overset{l}{\longrightarrow}^{n} x$ for some $n \in \mathbb{N}$. Also there are $m \in \mathbb{N}$ and x, x_i, s in S such that $x_1^m + sab = sab, x_{i+1}^m + sx_i = sx_i(i = 1, 2, ..., n-2), x^m + sx_{n-1} = sx_{n-1}$. Now $sab \in \Lambda(sa)$ implies that $sa \overset{l}{\longrightarrow}^{k} sab$ for some $n \in \mathbb{N}$. Again, there are $r \in \mathbb{N}$ and y_i, u in S such that

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$y_1^r + usa = usa, y_{i+1}^r + uy_i = uy_i (i = 1, 2, ..., k-2), (sab)^r + uy_{k-1} = uy_{k-1}$. Now one has $x_1^{mr} + (sab)^r = (sab)^r$. From this one gets $x_1^{mr} + uy_{k-1} = uy_{k-1}$. Thus we have $y_{k-1} \xrightarrow{l} x_1$. Also $a \xrightarrow{l} y_1 \xrightarrow{l} y_2 ... \xrightarrow{l} y_{k-1}$ and $x_1 \xrightarrow{l} x_2 ... \xrightarrow{l} x_{n-1} \longrightarrow x$. By transitivity of $\xrightarrow{l}^{\infty}, a \xrightarrow{l}^{\infty} x$, that is, $x \in \Lambda(a)$. This yields $\Lambda(ab) \subseteq \Lambda(a)$. Also $\Lambda(ab) \subseteq \Lambda(b)$ is clear. Thus $\Lambda(ab) \subseteq \Lambda(a) \cap \Lambda(b)$. The opposite inclusion is easy to check. Consequently, $\Lambda(ab) = \Lambda(a) \cap \Lambda(b)$.
$(2) \Rightarrow (6)$ Follows easily since $\Lambda(ab) \subseteq \Lambda(a)$. □

Next, in the following two lemmas we find the conditions which make the relation $\xrightarrow{l}^{n}$ transitive on S that plays a crucial role in characterizing the semirings which are distributive lattices of λ_n-simple semirings and is presented in Theorem 3.7.

Lemma 3.5. *Let S be a semiring and $n \in \mathbb{N}$ such that $\Lambda_n(a) \subseteq \Lambda_n(a^2)$. Then $\xrightarrow{l}^{n}$ is transitive on S.*

Proof. Let $a \xrightarrow{l}^{n+1} b$. Then there is $x \in S$ such that $a \xrightarrow{l} x \xrightarrow{l}^{n} b$, and so repeated application of the hypothesis one can find that $x^{2^r} \xrightarrow{l}^{n} b$ for every $r \in \mathbb{N}$. Also there exist $k \in \mathbb{N}$ and $s \in S$ such that $x^k + sa = sa$, by Lemma 3.1. Let $y \in S$ such that $x^{2^r} \longrightarrow y \longrightarrow^{n-1} b$, if $n \geq 2$, and $y = b$ if $n = 1$. The there are $m \in \mathbb{N}$ and $t \in S$ such that $y^m + tx^{2^r} = tx^{2^r}$(assuming $2^r > k$), i.e, $y^m + ua = ua$ for $u = ts$. Hence $a \xrightarrow{l} y$, which implies $a \xrightarrow{l}^{n} b$, i.e, $\xrightarrow{l}^{n+1} \subseteq \xrightarrow{l}^{n}$, and so $\xrightarrow{l}^{n} = \xrightarrow{l}^{n+1} = \xrightarrow{l}^{\infty}$. Thus $\xrightarrow{l}^{n}$ is transitive. □

Lemma 3.6. *Let $n \in \mathbb{N}$. Then the following are equivalent on a semiring S:*

(1) *for all $a \in S$, $a\lambda_n a^2$;*
(2) *for all $a, b \in S, a \xrightarrow{l}^{n} b \Rightarrow a^2 \xrightarrow{l}^{n} b$.*

Proof. $(1) \Rightarrow (2)$ Trivial.
$(2) \Rightarrow (1)$ Let $x \in \Lambda_n(a)$. So $a \xrightarrow{l}^{n} x$, and this implies $a^2 \xrightarrow{l}^{n} x$, i.e. $x \in \Lambda(a^2)$. Thus $\Lambda_n(a) \subseteq \Lambda_n(a^2)$. Conversely, let $y \in \Lambda_n(a^2)$. So $a^2 \xrightarrow{l}^{n} y$. Then by $a \xrightarrow{l}^{n} a^2$ and Lemma 3.5, we get $a \xrightarrow{l}^{n} y$, i.e. $y \in \Lambda_n(a)$. Thus $a\lambda_n a^2$. □

Theorem 3.7. *Let $n \in \mathbb{N}$. Then the following conditions are equivalent on a semiring S:*

(1) *S is a distributive lattice of λ_n-simple subsemirings;*
(2) *for all $a, b \in S, a\lambda_n a^2$ and $a \xrightarrow{l}^{n} ab$;*
(3) *for every $a \in S, \Lambda_n(a)$ is the least completely semiprime k-ideal of S containing a;*
(4) *$\lambda_n = \eta$, the least distributive lattice congruence on S.*

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Proof. (1) $\Rightarrow$ (2) Let S be a distributive lattice $\mathcal{D}$ of λ_n-simple subsemirings $S_i; i \in \mathcal{D}$. Assume $a, b \in S$ such that $a \in S_i, b \in S_j; i, j \in \mathcal{D}$. Then $a, a^2 \in S_i$ implies $a\lambda_n a^2$ in S_i, since S_i is λ_n-simple, i.e. $a\lambda_n a^2$ in S. Also $ab, ba \in S_{ij}$, so $ba \overset{l}{\longrightarrow}{}^{n} ab$ in S_{ij}, by Lemma 3.3. Now $a \overset{l}{\longrightarrow} ba$ and $ba \overset{l}{\longrightarrow}{}^{n} ab$ yield $a \overset{l}{\longrightarrow}{}^{n} ab$, by Lemma 3.5.
(2) $\Rightarrow$ (3) By Lemmas 3.5 and 3.6, $\overset{l}{\longrightarrow}{}^{n}$ is transitive on S and so $\Lambda_n(a) = \Lambda(a)$, which is the least completely semiprime k-ideal of S containing a, by Theorem 3.4.
(3) $\Rightarrow$ (4) Since $\Lambda_n(a)$ is the least completely semiprime k-ideal of S, $\Lambda_n(a) = \Sigma(a)$ and so $\lambda_n = \eta$.
(4) $\Rightarrow$ (1) Follows from Theorems 2.5 and 2.7. □

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